Three positive solutions of a nonlinear Dirichlet problem with competing power nonlinearities

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January 19, 2015

Abstract

This paper studies a nonlinear Dirichlet problem for the p-Laplacian operator with nonlinearity consisting of power components. The problem under consideration can be though of as a perturbation of the Ambrosetti-Brezis-Cerami problem with concave-convex nonlinearity. The combined effect of power components in the perturbed nonlinearity allows to establish a higher order multiplicity of positive solutions. We study properties of the perturbed energy functional and prove the existence of three positive solutions to the problem at hand.

1 Introduction

In this paper, we study the following nonlinear Dirichlet problem:

$$\begin{cases} -\Delta_p u = \lambda \left(|u|^{\alpha-2}u - |u|^{\gamma-2}u\right) + |u|^{\beta-2}u & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain, λ is a real parameter, and $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator. Problem (1) is studied under the following hypothesis on the exponents:

(H0) $1 < \alpha < p < \beta < \gamma < p^*$,

where p^* is the critical Sobolev exponent defined by

$$p^* := \begin{cases} pN/(N-p) & \text{if } p < N, \\ \infty & \text{if } p \geqslant N. \end{cases}$$

Our goal is to show that problem (1) has at least three positive solutions for any $\lambda \in (0, \lambda^*)$ where λ^* is some positive real number.

Multiplicity of positive solutions to nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f_\lambda(\cdot, u) & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

has been extensively studied by many authors. In [1], problem (2) was studied with $p = 2$, bounded Ω, and concave-convex nonlinearity $f_\lambda(x,u) = \lambda|u|^{\alpha-2}u + |u|^{\beta-2}u$, $1 < \alpha < p < \beta \leqslant p^*$, where the existence of two positive solutions for $\lambda \in (0, \lambda^*)$, $\lambda^* > 0$, has been proved. This result was generalized for $p \geqslant 2$ and/or other nonlinear operators, and for more general nonlinearities essentially preserving the concave-convex structure (cf. [2, 3, 4, 5]). It is known that, at least for the case where Ω is the unit ball, two positive solutions for the concave-convex problem considered in [1] is the maximum one can expect (cf. [6]).

Some recent papers that study higher order multiplicity of positive solutions to nonlinear problems are [7, 8, 9]. In [7], a one-dimensional problem (2) with $p = 2$ and $f_\lambda(x,u) = \lambda g(u)$ has been considered for $\Omega = (-1, 1)$. Under the assumption that g is concave on $(0, \gamma)$ and convex on (γ, ∞), $\gamma > 0$, as well as other assumptions such as g having a unique positive zero and $\lim_{u\to\infty} g(u)/u = \infty$, the existence of

three positive solutions has been proved for $\lambda \in (\lambda_*, \lambda^*)$, $\lambda^* > \lambda_* > 0$. In [8], problem (2) was considered for $p > N$ and $f_\lambda(x,u) = \lambda a(x)u^{-\gamma} + \lambda g(x,u)$, $\gamma > 0$. Under some assumptions on the coefficients and the exponent γ, the existence of three positive solutions has been proved for λ belonging to an open subinterval of $(0,\infty)$. Finally, in [9], a one-dimensional problem (2) (the authors considered a more general form of the differential operator) with $f_\lambda(x,u) = \lambda a(x)g(u)$ and $\Omega = (0,1)$ has been considered. Under various assumptions, including $g(u) \leqslant u^{p-1}$ in a positive neighborhood of 0, the existence of three positive solutions has been shown for λ belonging to an open subinterval of $(0,\infty)$.

Problem (1), which we study in this paper, is different from that in [1] in the presence of the dominating negative term $-\lambda|u|^{\gamma-2}u$ which is responsible for producing an extra positive solution. This leads to the following main result of this paper.

Theorem 1.1. *Under hypothesis (H0), there exists an $\lambda^* > 0$ such that for any $\lambda \in (0,\lambda^*)$, problem* (1) *has three distinct positive solutions $u_1, u_2, u_3 \in C^{1,\mu}(\bar{\Omega})$, $0 < \mu < 1$, such that*

$$\max(J_\lambda(u_1), J_\lambda(u_3)) < 0 < J_\lambda(u_2)$$

where

$$J_\lambda(v) = \frac{1}{p}\int_\Omega |\nabla v|^p - \frac{\lambda}{\alpha}\int_\Omega |u|^\alpha - \frac{1}{\beta}\int_\Omega |u|^\beta + \frac{\lambda}{\gamma}\int_\Omega |u|^\gamma$$

is an energy functional for problem (1).

Many powerful methods exist in nonlinear analysis that help study the multiplicity of solutions to nonlinear differential equations. These methods include Morse theory, mountain pass lemma, fixed point theorems, and the Pohozaev fibering method to name a few (cf. [10, 11, 12, 13] and references therein). The fibering method is especially useful when the nonlinearity contains polynomial components and will serve as the underlying method in this paper. For its applications to multiplicity of solutions to nonlinear equations and systems, cf. [14, 15, 2, 16] and references therein.

This paper is organized as follows. Section 2 contains some known results to be used in the paper. Section 3 studies properties of the energy functional, J_λ, necessary to prove Theorem 1.1. Sections 4–6 study three optimization problems, each leading to a positive solution of problem (1). Finally, Section 7 is dedicated to proving our main result, Theorem 1.1.

2 Preliminaries

In this section, we list two known results that help studying critical points of a functional defined on a Banach space. Throughout this section, differentiability always means Fréchet differentiability.

Let $(X, \|\cdot\|)$ be a Banach space, and let B_ρ and S_ρ be respectively its closed ball and sphere of radius ρ. We are concerned with critical points of a differentiable functional $J\colon X \to \mathbf{R}$. In other words, we want to study the equation

$$DJ(u) = 0, \qquad u \in X. \tag{3}$$

2.1 The fibering method

In this subsection, we present the Fibering Method of S. Pohozaev [17, 14, 13]. We will state its special form, the *spherical fibering*, which is most suitable for our problem.

Suppose that the norm $\|\cdot\|$ of the Banach space X is differentiable away from the origin and that

$$J = \frac{1}{p}\|\cdot\|^p + R$$

where $p > 1$ and $R\colon X \to \mathbf{R}$ is a differentiable functional.

Instead of studying critical points of the functional J directly, constrained critical points of the fibering functional

$$I(v) = J(t(v)v)$$

where $t = t(v)$ is a solution of

$$\frac{\partial}{\partial t}J(tv) = 0$$

are studied on S_1. Under certain assumptions, a constrained critical point v_* of I corresponds to a critical point $u_* = t(v_*)v_*$ of the original functional J.

These constrained critical points typically arise as extrema of I on S_1. But because S_1 is not weakly closed, the limit of a maximizing or minimizing sequence, if it exists, may no longer be in S_1. There is a regularity theorem that helps circumvent this difficulty. It reduces the problem to studying constrained critical points of an extension of I to the unit ball. For reflexive X, this new problem is often more tractable because of the weak compactness of B_1.

A rigorous formulation of the (spherical) fibering method combined with the regularity theorem is presented below.

Theorem 2.1. *Let $\mathscr{U}$ be an open subset of X with $\mathscr{U} \cap S_1 \neq \varnothing$. Suppose that the equation*

$$t^{p-1} + DR(tv)v = 0 \tag{4}$$

has a solution $t\colon B_1 \cap \mathscr{U} \to [0, \infty)$ that is differentiable on $(B_1 \cap \mathscr{U}) \setminus \{0\}$. Consider the functional $I\colon (B_1 \cap \mathscr{U}) \setminus \{0\} \to \mathbf{R}$ defined by

$$I(v) = \frac{1}{p} t(v)^p + R(t(v)v)$$

and let

$$\mathscr{M} := \{v \in B_1 \cap \mathscr{U} \mid t(v) \neq 0\}.$$

Then for any critical point $v_* \in \mathscr{M}$ of I on $B_1 \cap \mathscr{U}$,

(a) *$v_* \in S_1$;*

(b) *$u_* = t(v_*)v_*$ is a critical point of J, provided that $DH(v_*)v_* \neq 0$ where $H := \|\cdot\|$.*

2.2 Mountain pass theorem

Another result in the critical point theory that we will use is the Mountain Pass Theorem by A. Ambrosetti and P. Rabinowitz [18]. For that theorem to be applied, a certain compactness condition needs to be satisfied, which is introduced is the definitions below.

Definition 2.1. Given $c \in \mathbf{R}$, a sequence $(u_n) \subset X$ is called a Palais-Smale sequence at level c ($(\mathrm{PS})_c$-sequence, in short) for the differentiable functional $J\colon X \to \mathbf{R}$ if

$$J(u_n) \to c \qquad \text{and} \qquad DJ(u_n) \to 0 \text{ in } X^*.$$

Definition 2.2. A differentiable functional $J\colon X \to \mathbf{R}$ is said to satisfy the Palais-Smale condition if given $c \in \mathbf{R}$, every $(\mathrm{PS})_c$-sequence for J contains a (strongly) convergent subsequence.

The theorem is question is presented below.

Theorem 2.2. *Let $J \in C^1(X, \mathbf{R})$ satisfy the Palais-Smale condition and let*

$$\max(J(0), J(w)) < \inf_{S_\rho} J$$

for some $\|w\| > \rho > 0$. Then J has a critical point u_* such that

$$J(u_*) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

where

$$\Gamma := \{\gamma \in C([0,1], X) \mid \gamma(0) = 0 \text{ and } \gamma(1) = w\}.$$

3 Some properties of the energy functional

It will be convenient for us to represent J_λ as

$$J_\lambda(v) = \frac{1}{p}\int_\Omega |\nabla v|^p - \frac{\lambda}{\alpha}A(v) - \frac{1}{\beta}B(v) + \frac{\lambda}{\gamma}C(v)$$

where

$$A(u) := \int_\Omega |u|^\alpha, \qquad B(u) := \int_\Omega |u|^\beta, \qquad C(u) := \int_\Omega |u|^\gamma.$$

Lemma 3.1. *The energy functional J_λ is coercive for any $\lambda > 0$.*

Proof. It suffices to show that the functional

$$R_\lambda(v) := -\frac{\lambda}{\alpha}A(v) - \frac{1}{\beta}B(v) + \frac{\lambda}{\gamma}C(v) \tag{5}$$

is bounded from below on $W_0^{1,p}(\Omega)$. It is readily seen that

$$R_\lambda(v) \geqslant \frac{\lambda}{\gamma}\|v\|_{L^\gamma}^\gamma - c_1\|v\|_{L^\gamma}^\beta - \lambda c_2\|v\|_{L^\gamma}^\alpha$$

for some constants c_1 and c_2 independent of v. We complete the proof by noticing that the function

$$s \mapsto \frac{\lambda}{\gamma}s^\gamma - c_1 s^\beta - \lambda c_2 s^\alpha$$

is bounded from below on $[0,\infty)$.

$\square$

Let

$$\hat{J}_\lambda(t,v) := \frac{1}{p}t^p + R_\lambda(tv) \qquad \text{and} \qquad \Phi_\lambda(t,v) := t^{-\alpha}\frac{\partial}{\partial t}\hat{J}_\lambda(t,v).$$

Clearly, $\hat{J}_\lambda(t,v) = J_\lambda(tv)$ for $v \in S_1$. Consider the equation

$$\Phi_\lambda(t,v) = t^{p-\alpha} - t^{\beta-\alpha}B(v) + \lambda t^{\gamma-\alpha}C(v) - \lambda A(v) = 0, \qquad t > 0. \tag{6}$$

Because of our assumption of the exponents, (H0), this equation has at most three solutions. Set

$$\mathscr{U}_\lambda := \{v \in W_0^{1,p}(\Omega) \mid \text{ equation (6) has three solutions in } t\}.$$

Given $v \in W_0^{1,p}(\Omega)$, the i^{th} solution of (6) will be denoted $t_i(v) = t_{i,\lambda}(v)$, where the symbol λ will be dropped for simplicity.

Lemma 3.2. *There exists an $\Lambda > 0$ such that for any $\lambda \in (0,\Lambda)$, the following hold.*

(a) $\mathscr{U}_\lambda \cap S_1 \neq \varnothing$.

(b) $\mathscr{U}_\lambda$ *is open in* $W_0^{1,p}(\Omega)$.

(c) *For any $v \in \mathscr{U}_\lambda \cap B_1$,*

$$\hat{J}_\lambda(t_1(v),v) < 0 < \hat{J}_\lambda(t_2(v),v).$$

There exists an $w \in \mathscr{U}_\lambda \cap S_1$ such that

$$J_\lambda(t_3(w)w) < 0.$$

Proof. (a): By the Rellich-Kondrachov theorem, there is an $v^* \in S_1$ such that

$$B(v^*) = \max_{v \in S_1} B(v).$$

Choose $\Lambda > 0$ so that

$$\max_{t>0} \left[t^{p-\alpha} - t^{\beta-\alpha} B(v^*)\right] > \Lambda \max_{v \in S_1} A(v) \tag{7}$$

and

$$\min_{t>0} \left[t^{p-\alpha} - t^{\beta-\alpha} B(v^*) + \Lambda\, t^{\gamma-\alpha} C(v^*)\right] < 0.$$

Then it is clear that $v^* \in \mathscr{U}_\lambda$ for all $\lambda \in (0, \Lambda)$.

(b): Consider the map

$$T_\lambda(t,v) := \left(\Phi_\lambda(t,v), \frac{\partial}{\partial t}\Phi_\lambda(t,v)\right).$$

We will write $T_\lambda(t,v) > 0$ (resp., $T_\lambda(t,v) < 0$) to mean that the two components of $T_\lambda(t,v)$ are strictly positive (resp., strictly negative).

It is readily seen that $v \in \mathscr{U}_\lambda$ if and only if

$$T_\lambda(s_1, v) > 0 \qquad \text{and} \qquad T_\lambda(s_2, v) < 0$$

for some $0 < s_1 < s_2 < \infty$. Now fix any $v_0 \in \mathscr{U}_\lambda$ and choose $0 < s_1^0 < s_2^0 < \infty$ with

$$T_\lambda(s_1^0, v_0) > 0 \qquad \text{and} \qquad T_\lambda(s_2^0, v_0) < 0.$$

Since $T_\lambda(t, \cdot)$ is continuous on $W_0^{1,p}(\Omega)$ for any $t > 0$, there is a neighborhood $\mathscr{N}_{v_0}$ of v_0 in $W_0^{1,p}(\Omega)$ such that

$$T_\lambda(s_1^0, v) > 0 \qquad \text{and} \qquad T_\lambda(s_2^0, v) < 0 \qquad \text{for all } v \in \mathscr{N}_{v_0}.$$

This implies that $\mathscr{N}_{v_0} \subset \mathscr{U}_\lambda$. Since v_0 was arbitrary, we conclude that $\mathscr{U}_\lambda$ is open.

(c): Reduce Λ, if necessary, so that

$$\max_{t>0} \left[\frac{t^{p-\alpha}}{p} - \frac{t^{\beta-\alpha}}{\beta} B(v^*)\right] > \frac{\Lambda}{\alpha} \max_{v \in S_1} A(v) \tag{8}$$

and

$$\min_{t>0} \left[\frac{t^p}{p} - \frac{t^\beta}{\beta} B(v^*) + \Lambda \frac{t^\gamma}{\gamma} C(v^*)\right] < 0. \tag{9}$$

Fix any $v \in \mathscr{U}_\lambda \cap B_1$. Then the inequality

$$\hat{J}_\lambda(t_1(v), v) < 0$$

follows from the fact that $\alpha < \min(p, \beta, \gamma)$. It is clear that

$$t \mapsto \hat{J}_\lambda(t, v) \qquad \text{is increasing on } [t_1(v), t_2(v)]. \tag{10}$$

By (7), for any $v \in B_1 \setminus \{0\}$, the equation

$$\tilde{\Phi}_\lambda(t,v) := t^{p-\alpha} - t^{\beta-\alpha} B(v) - \lambda A(v) = 0$$

has exactly two solutions, $0 < \tilde{t}_1(v) < \tilde{t}_2(v) < \infty$.

Let us show that

$$t_1(v) < \tilde{t}_1(v) < \tilde{t}_2(v) < t_2(v). \tag{11}$$

Since $\Phi_\lambda(t,v) > \tilde{\Phi}_\lambda(t,v)$ for all $t > 0$,

$$\Phi_\lambda(\cdot, v) > 0 \qquad \text{on } [\tilde{t}_1(v), \tilde{t}_2(v)].$$

Therefore, either $[\tilde{t}_1(v), \tilde{t}_2(v)] \subset (t_1(v), t_2(v))$ or $[\tilde{t}_1(v), \tilde{t}_2(v)] \subset (t_3(v), \infty)$. We want to verify that the second option is impossible. Indeed, since $\frac{\partial}{\partial t}\Phi_\lambda(t,v) > \frac{\partial}{\partial t}\tilde{\Phi}_\lambda(t,v)$ for $t > 0$ and $\frac{\partial}{\partial t}\tilde{\Phi}_\lambda(t,v) > 0$ on $(0, \tilde{t}_1(v)]$,

$$\Phi_\lambda(\cdot, v) \qquad \text{is increasing on } (0, \tilde{t}_1(v)].$$

This implies that $\tilde{t}_1(v) < t_2(v)$, finishing the proof of (11).

Entertaining (10), (11), we deduce that

$$\hat{J}_\lambda(t_2(v), v) > \hat{J}_\lambda(\tilde{t}_2(v), v) \geqslant H(\tilde{t}_2(v), v)$$

where

$$H(t, v) := \frac{t^p}{p} - \lambda \frac{t^\alpha}{\alpha} A(v) - \frac{t^\beta}{\beta} B(v).$$

But by (8),

$$H(\tilde{t}_2(v), v) = \max_{t>0} H(t, v) > 0,$$

yielding that $\hat{J}_\lambda(t_2(v), v) > 0$.

Finally, take $w = v^*$. The minimum point for the left hand side of (9) is bounded from below by a positive constant as $\Lambda \downarrow 0$, whereas the first zero of $t \mapsto J_\lambda(tw)$ is $o(1)$ as $\lambda \downarrow 0$. This means that we can decrease Λ, if necessary, so that the third critical value of $t \mapsto J_\lambda(tw)$ is negative or, equivalently, $J_\lambda(t_3(w)w) < 0$.

□

4 First critical point

Let $\Lambda > 0$ be as in part (b) of Lemma 3.2. In view of inequality (7), equation (6) has the minimal solution $t = t_1(v)$ for any $v \in B_1 \setminus \{0\}$. Moreover,

$$\left.\frac{\partial}{\partial t}\right|_{t=t_1(v)} \Phi_\lambda(t, v) > 0. \tag{12}$$

Lemma 4.1. *The functional t_1 is continuously differentiable and weakly continuous on $B_1 \setminus \{0\}$.*

Proof. The statement of the lemma follows from (12), the Implicit Function Theorem, and the weak continuity of A, B, and C.

□

Define the functional $I_1 \colon B_1 \setminus \{0\} \to \mathbf{R}$ by

$$I_1(v) := \hat{J}_\lambda(t_1(v), v) < 0.$$

Theorem 4.1. *For any $\lambda \in (0, \Lambda)$, there is a nonnegative $v_* \in S_1$ such that $u_* = t_1(v_*)v_*$ is a critical point of J_λ with $J_\lambda(u_*) < 0$.*

Proof. Consider the problem

$$c_1(\lambda) := \inf_{v \in B_1 \setminus \{0\}} I_1(v). \tag{13}$$

Let (v_n) be its minimizing sequence, which we may assume to be nonnegative. According to Lemma 3.1, $t_{1,n} := t_1(v_n)$ is a bounded sequence. Therefore, without loss of generality we can assume that

$$\begin{aligned} v_n &\rightharpoonup v_* \qquad \text{weakly in } W^{1,p}(\Omega), \\ t_{1,n} &\to t_* \end{aligned}$$

for some nonnegative $v_* \in B_1$ and $0 \leqslant t_* < \infty$.

Since $c_1(\lambda) < 0$, we must have $0 < t_* < \infty$. In particular, equation (6) implies that $v_* \not\equiv 0$. By Lemma 4.1 and the weak closedness of B_1, the infimum in (13) is attained at v_*.

Entertaining Theorem 2.1 and Lemma 4.1, we conclude that $v_* \in S_1$ and that $u_* = t_1(v_*)v_*$ is a critical point of J_λ. Since $v_* \in S_1$, $J_\lambda(u_*) = I_1(v_*) < 0$.

□

5 Second critical point

Consider the continuously differentiable functional $\bar{J}_\lambda \colon W_0^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$\bar{J}_\lambda(v) := \frac{1}{p}\int_\Omega |\nabla v|^p + R_\lambda(v^+)$$

where $v^+ := \max(v, 0)$.

Lemma 5.1. *The following statements are true.*

(a) *$\bar{J}_\lambda$ satisfies the Palais-Smale condition for any $\lambda > 0$.*

(b) *There exists an $\rho > 0$ such that for any $\lambda \in (0, \Lambda)$,*

$$\max\left(\inf_{B_\rho} \bar{J}_\lambda, \inf_{W_0^{1,p}\setminus B_\rho} \bar{J}_\lambda\right) < 0 < \inf_{S_\rho} \bar{J}_\lambda.$$

Proof. (a): Fix any $c \in \mathbf{R}$ and a $(\mathrm{PS})_c$-sequence (u_n) for $\bar{J}_\lambda$:

$$\bar{J}_\lambda(u_n) \to c, \tag{14}$$

$$D\bar{J}_\lambda(u_n) \to 0 \qquad \text{in } W^{-1,p'}(\Omega). \tag{15}$$

Mimicking the proof of Lemma 3.1, one verifies that the functional $\bar{J}_\lambda$ is coercive. Therefore (14) implies that (u_n) is bounded.

Since $\Delta_p \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is a homeomorphism, we can write

$$(-\Delta_p)^{-1} D\bar{J}_\lambda(u_n) = u_n + (-\Delta_p)^{-1} DR_\lambda(u_n^+).$$

By our assumptions (H0) on the exponents, $DR_\lambda \colon W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is a compact operator, and hence so is the operator $K_\lambda := (-\Delta_p)^{-1} DR_\lambda \colon W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$. Taking into account (15), we obtain

$$o(1) = u_n + K_\lambda(u_n^+).$$

Since (u_n) is bounded and K_λ is compact, we conclude that (u_n) contains a (strongly) convergent subsequence in $W_0^{1,p}(\Omega)$.

(b): Denote by $\rho > 0$ the value of t where the left hand side of (8) attains its maximum. It is clear that

$$\inf_{B_\rho} \bar{J}_\lambda < 0 < \inf_{S_\rho} \bar{J}_\lambda.$$

Since the value of t where the left hand side of (9) attains its minimum is $> \rho$, we also deduce that

$$\inf_{W_0^{1,p}(\Omega)\setminus B_\rho} \bar{J}_\lambda < 0.$$

$\square$

Theorem 5.1. *For any $\lambda \in (0, \Lambda)$, the functional J_λ has a critical point $u_* \geqslant 0$ with $J_\lambda(u_*) > 0$.*

Proof. According to statement (b) of Lemma 5.1, there is an $\rho > 0$ and an $w \in W_0^{1,p}(\Omega) \setminus B_\rho$ such that

$$\max(\bar{J}_\lambda(0), \bar{J}_\lambda(w)) = 0 < \inf_{S_\rho} \bar{J}_\lambda.$$

Since, by statement (a) of the same Lemma, $\bar{J}_\lambda$ satisfies the Palais-Smale condition, we conclude from the Mountain Pass Theorem 2.2 that $\bar{J}_\lambda$ has a critical point u_* such that

$$\bar{J}_\lambda(u_*) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \bar{J}_\lambda(\gamma(t)) > 0$$

where

$$\Gamma := \{\gamma \in C([0,1], W_0^{1,p}(\Omega)) \mid \gamma(0) = 0 \text{ and } \gamma(1) = w\}.$$

To finish the proof, it suffices to show that $u_* \geqslant 0$ because then u_* will be a critical point of the original functional J_λ. Since

$$0 = D\bar{J}_\lambda(u_*)u_*^- = \int_\Omega |\nabla u_*|^{p-2}\nabla u_* \cdot \nabla u_*^- = \int_\Omega |\nabla u_*^-|^p$$

and $u_* \in W_0^{1,p}(\Omega)$, we must have $u_*^- = 0$ or, equivalently, that $u_* \geqslant 0$.

□

6 Third critical point

Define the functional $I_3 \colon \mathscr{U}_\lambda \to \mathbf{R}$ by

$$I_3(v) := \hat{J}_\lambda(t_3(v), v) = \min_{t > t_2(v)} \hat{J}_\lambda(t, v).$$

Lemma 6.1. *The functional t_3 is continuously differentiable and weakly continuous on $\mathscr{U}_\lambda$.*

Proof. The statement of the lemma follows from the inequality

$$\left.\frac{\partial}{\partial t}\right|_{t=t_3(v)} \Phi_\lambda(t, v) > 0, \qquad v \in \mathscr{U}_\lambda$$

the Implicit Function Theorem, and the weak continuity of A, B, and C.

□

Theorem 6.1. *For any $\lambda \in (0, \Lambda)$, there is a nonnegative $v_* \in S_1 \cap \mathscr{U}_\lambda$ such that $u_* = t_3(v_*)v_*$ is a critical point of J_λ with $J_\lambda(u_*) < 0$.*

Proof. By statement (c) of Lemma 3.2,

$$c_3(\lambda) := \inf_{v \in \mathscr{U}_\lambda \cap B_1} I_3(v) < 0. \tag{16}$$

Let (v_n) be a minimizing sequence for problem (16) which we may assume to be nonnegative. Let $t_{3,n} := t_3(v_n)$. Then we can assume that

$$v_n \rightharpoonup v_* \qquad \text{weakly in } W_0^{1,p}(\Omega), \tag{17}$$

$$t_{3,n} \to t_3 \tag{18}$$

for some nonnegative $v_* \in B_1$ and $0 \leqslant t_3 \leqslant \infty$.

Since, by Lemma 3.1, J_λ is coercive and $I_3(v_n) \geqslant J_\lambda(t_{3,n}v_n)$, we must have $t_3 < \infty$. Entertaining (16), we also obtain that $0 < t_3$. In other words, $0 < t_3 < \infty$. Taking into account equation (6), we infer that $v_* \not\equiv 0$.

Let us show that $v_* \in \mathscr{U}_\lambda$. Let $t_{2,n} := t_2(v_n)$. Without loss of generality, we may assume that

$$t_{2,n} \to t_2$$

for some $0 \leqslant t_2 \leqslant t_3$. Since $v_* \not\equiv 0$, we deduce from (6) that $t_2 \neq 0$ and hence

$$0 < t_2 \leqslant t_3.$$

It is also clear that both t_2 and t_3 are solutions to equation (6) for $v = v_*$:

$$\Phi_\lambda(t_i, v_*) = 0, \qquad i = 2, 3.$$

By statement (c) of Lemma 3.2,

$$0 < \frac{t_{2,n}^p}{p} + R_\lambda(t_{2,n}v_n), \qquad n \geqslant 1,$$

yielding that

$$0 \leqslant \frac{t_2^p}{p} + R_\lambda(t_2 v_*). \tag{19}$$

Since

$$\frac{t_3^p}{p} + R_\lambda(t_3 v_*) = c_3(\lambda) < 0,$$

we deduce that

$$0 < t_2 < t_3.$$

Since α is the smallest exponent, we deduce from (19) that equation (6) for $v = v^*$ has a third solution, $t_1 \in (0, t_2)$, satisfying

$$\frac{t_1^p}{p} + R_\lambda(t_1 v_*) < 0.$$

Therefore, equation (6) for $v = v_*$ has three solutions, yielding that $v_* \in \mathscr{U}_\lambda$. So, we deduce that the infimum in (16) is attained at v_*.

Applying Theorem 2.1 and Lemma 6.1, we conclude that $v_* \in S_1$ and that $u_* = t_3(v_*)v_*$ is a critical point of J_λ. Since $v_* \in S_1$, $J_\lambda(u_*) = I_3(v_*) < 0$.

□

7 Proof of the main result

In this section, we prove the main result of this paper, Theorem 1.1. We will also need the following regularity result (cf. [19]).

Theorem 7.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with C^2 boundary and let $u \in W_0^{1,p}(\Omega)$ be a weak solution of the problem*

$$\begin{cases} -\Delta_p u = f(\cdot, u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

where $f\colon \Omega \times \mathbf{R}$ is a Carathéodory function that satisfies the inequality

$$|f(x,u)| \leqslant c(1 + |u|^{m-1}) \qquad \text{for a.e. } x \in \Omega \text{ and all } u \in \mathbf{R},$$

where $m = p^*$ if $p < N$ and $m > 1$ if $p \geqslant N$. Then $u \in C^{1,\mu}(\bar{\Omega})$ for some $\mu \in (0,1)$.

We now move on to proving the main result.

Proof of Theorem 1.1. Denote by λ^* the number Λ in Lemma 3.2 and fix any $\lambda \in (0, \lambda^*)$.

Let u_1, u_2, u_3 be functions u_* in Theorem 4.1, Theorem 5.1, and Theorem 6.1, respectively. Since they are critical points of the energy functional, they are weak solutions of problem (1). Since $J_\lambda(u_i) < 0 < J_\lambda(u_2)$ for $i = 1, 3$, $u_2 \neq u_1$ and $u_2 \neq u_3$. Let us show that $u_1 \neq u_3$. If this was not the case, we would have $v_1 = v_3$ and $t_1(v_1) = t_3(v_3)$. But then that would imply $t_1(v_3) = t_1(v_1) = t_3(v_3)$, contradicting the definition of $\mathscr{U}_\lambda$.

Applying Theorem 7.1, we infer that $u_1, u_2, u_3 \in C^{1,\mu}(\bar{\Omega})$ for some $\mu \in (0,1)$, which completes our proof.

□

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